Worksheet 8 Solutions

1) a) $\int \frac{1}{(x+1)^2+9}\,dx$ Let $u=x+1$, $du=dx$

$$= \int \frac{1}{u^2+9}\,du$$

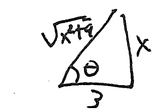
$$= \frac{1}{3}\arctan\left(\frac{u}{3}\right)+C$$

$$= \boxed{\frac{1}{3}\arctan\left(\frac{x+1}{3}\right)+C}$$

b) $\int \cos^2(x)\,dx = \frac{1}{2}\int 1+\cos(2x)\,dx$

$$= \boxed{\frac{1}{2}\left(x+\frac{1}{2}\sin(2x)\right)+C}$$

c) $\int \frac{1}{x\sqrt{x^2+9}}\,dx$ Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$



$$= \int \frac{1}{3\tan\theta\ 3\sec\theta}\,3\sec^2\theta\,d\theta$$

$$= \frac{1}{3}\int \frac{\sec\theta}{\tan\theta}\,d\theta = \frac{1}{3}\int \frac{1}{\sin\theta}\,d\theta = \frac{1}{3}\int \csc\theta\cdot\frac{\csc\theta+\cot\theta}{\csc\theta+\cot\theta}\,d\theta$$

$u = \csc\theta+\cot\theta$

$du = -\csc\theta\cdot\cot\theta-\csc^2\theta\,d\theta$

$$= \frac{1}{3}\int -\frac{du}{u}$$

$$= -\frac{1}{3}\ln\left|\csc\theta+\cot\theta\right|+C$$

$$= \boxed{-\frac{1}{3}\ln\left|\frac{\sqrt{x^2+9}}{x}+\frac{3}{x}\right|+C}$$

d) $\int_{0^-}^{-1} \frac{e^{1/x}}{x^3}\,dx = \lim_{R\to 0^-}\int_R^{-1}\frac{e^{1/x}}{x^3}\,dx$ $u=\frac{1}{x}$ $du=-\frac{1}{x^2}dx$ $x=-1 \to u=-1$, $x=R\to u=\frac{1}{R}$

$$= \lim_{R\to 0^-}\int_{1/R}^{-1} -u e^{u}\,du$$

$$= \lim_{R\to 0^-} -ue^u+e^u\Big|_{1/R}^{-1} \quad \text{(by parts)}$$

$$= \lim_{R\to 0^-}\left(e^{-1}+e^{-1}\right)-\left(\frac{-1}{R}e^{1/R}+e^{1/R}\right)$$

$$= \frac{2}{e}+\lim_{R\to 0^-}\frac{Re^{1/R}-e^{1/R}}{R}$$

$= \frac{2}{e}+0$ since $e^{1/R}\to$ "$e^{-\infty}$" $=0$ as $R\to 0^-$.

e) $\int x^3\sqrt{9-x^2}\,dx$ $u=9-x^2$, $du=-2x\,dx$, $x^2=9-u$

$$= \int x^2\sqrt{9-x^2}\,x\,dx$$

$$= -\frac{1}{2}\int (9-u)\sqrt{u}\,du$$

$$= -\frac{1}{2}\int u^{3/2}-9u^{1/2}\,du$$

$$= \frac{1}{2}\left[\frac{2}{5}u^{5/2}-6u^{3/2}\right]+C$$

$$= \boxed{\frac{1}{5}(9-x^2)^{5/2}-3(9-x^2)^{3/2}+C}$$

f) $\int_1^\infty xe^{-2x}\,dx = \lim_{R\to\infty}\int_1^R xe^{-2x}\,dx$

$$= \lim_{R\to\infty} -\frac{1}{2}xe^{-2x}\Big|_1^R+\frac{1}{2}\int_1^R e^{-2x}\,dx$$

$$= \lim_{R\to\infty} -\frac{1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}\Big|_1^R$$

$$= \lim_{R\to\infty} -\frac{1}{2}Re^{-2R}-\frac{1}{4}e^{-2R}+\frac{1}{2}e^{-2}+\frac{1}{4}e^{-2}$$

$$= 0+0+\frac{3}{4e^2} = \boxed{\frac{3}{4e^2}}$$

g) $\int \sin^2(x)\sin(2x)\cos(x)\,dx = 2\int \sin^3(x)\cos^2(x)\,dx = -2\int (1-u^2)u^2\,du$

$u = \cos(x)$

$du = \sin(x)\,dx$

$= -2\int u^2 - u^4\,du$

$= \boxed{\frac{2}{5}\cos^5(x) - \frac{2}{3}\cos^3(x) + C}$

h) $\int 6x^2 \frac{\cos(2x^3-5)}{\sin(2x^3-5)}\,dx \qquad u = \sin(2x^3-5)$

$du = \cos(2x^3-5)\cdot 6x^2\,dx$

$= \int \frac{1}{u}\,du$

$= \boxed{\ln|\sin(2x^3-5)| + C}$

i) $\int \frac{dx}{\sqrt{6-x^2}} = \frac{1}{\sqrt{6}}\int \sqrt{\frac{6}{6-x^2}}\,dx$

$= \frac{1}{\sqrt{6}}\int \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{6}}\right)^2}}\,dx$

$u = \frac{x}{\sqrt{6}}, \quad du = \frac{1}{\sqrt{6}}\,dx$

$= \int \frac{1}{\sqrt{1-u^2}}\,du$

$= \arcsin(u) + C$

$= \boxed{\arcsin\left(\frac{x}{\sqrt{6}}\right) + C}$

j) $\int_1^2 \frac{1}{x\ln(x)}\,dx = \lim_{t\to 1^+} \int_t^2 \frac{1}{x\ln(x)}\,dx \qquad u = \ln(x)$

$du = \frac{1}{x}\,dx$

• improper b/c

$\frac{1}{1\ln(1)} = \frac{1}{0}$

$= \lim_{t\to 1^+} \int \frac{1}{u}\,du$

$= \lim_{t\to 1^+} \ln|\ln(x)|\Big|_t^2$

$= \lim_{t\to 1^+} \ln|\ln(2)| - \ln|\ln(t)|$

$= \boxed{\infty} \qquad \downarrow \ln(0)\to\infty$

$\boxed{\text{So the integral diverges}}$

k) $\int \frac{x+3}{(x-6)(x-3)}\,dx$

$\frac{x+3}{(x-6)(x-3)} = \frac{A}{x-6} + \frac{B}{x-3} \Rightarrow x+3 = A(x-3) + B(x-6)$

$x+3 = (A+B)x - 3A - 6B$

So $A+B=1$ and $3 = -3A-6B$

$A = 1-B \quad \rightarrow \quad 3 = -3+3B-6B$

$6 = -3B$

$\boxed{B=-2}$

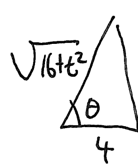
$\boxed{A=3}$

$= \int \frac{3}{x-6} - \frac{2}{x-3}\,dx$

$= \boxed{3\ln|x-6| - 2\ln|x-3| + C}$

l) $\int \sqrt{16+4x^2}\,dx \qquad t = 2x \cdot dt = 2dx$

$= \frac{1}{2}\int \sqrt{16+t^2}\,dt \qquad t = 4\tan\theta \quad dt = 4\sec^2\theta\,d\theta$



$= \frac{1}{2}\int 4\sec^3\theta\,d\theta$

$= 2\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right) + C$

$= \boxed{\frac{\sqrt{16+4x^2}}{4}\cdot\frac{2x}{4} + \ln\left|\frac{\sqrt{16+4x^2}}{4} + \frac{x}{2}\right| + C}$

m) $\int_0^2 \frac{x}{x-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{x}{x-1}\,dx + \lim_{s\to 1^+} \int_s^2 \frac{x}{x-1}\,dx$

• $x=1$ is a vertical asymptote, so this improper!

$= \lim_{t\to 1^-} \int_0^t 1+\frac{1}{x-1}\,dx$

(by division)

$= \lim_{t\to 1^-} x+\ln|x-1|\Big|_0^t$

$= \lim_{t\to 1^-} t+\ln|t-1|-0$

$= 1+-\infty$

$\doteq -\infty$

So $\underline{\int_0^2 \frac{x}{x-1}\,dx \text{ diverges!}}$ (we don't need to do the other part since this one diverged).

• If the first integral had converged, we would need to do the second as well.

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n) $\int \frac{3x^2+9x+8}{x^2(x+2)^2}\,dx$

$\frac{3x^2+9x+8}{x^2(x+2)^2} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+2}+\frac{D}{(x+2)^2}$

$3x^2+9x+8 = Ax(x+2)^2 + B(x+2)^2 + Cx^2(x+2) + Dx^2$

$x=0: \quad 8=4B \qquad \underline{B=2}$

$x=-2: \quad 12-18+8 = 4D$

$2=4D \qquad \underline{D=\frac{1}{2}}$

$x=1: \quad 3+9+8 = 9A+18+3C+\frac{1}{2}$

$\frac{3}{2} = 9A+3C$

$x=-1: \quad 3-9+8 = -A+2+C+\frac{1}{2}$

$-\frac{1}{2} = -A+C$

$C = A-\frac{1}{2}$

$\frac{3}{2} = 9A+3A-\frac{3}{2}$

$3 = 12A$

$\underline{A=\frac{1}{4}} \qquad \underline{C=-\frac{1}{4}}$

$= \int \frac{1}{4}\cdot\frac{1}{x}+\frac{2}{x^2}-\frac{1}{4}\cdot\frac{1}{x+2}+\frac{1}{2}\frac{1}{(x+2)^2}\,dx$

$= \boxed{\frac{1}{4}\ln|x| - \frac{2}{x} - \frac{1}{4}\ln|x+2| - \frac{1}{2(x+2)} + C}$

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o) $\int_0^{\pi} x^2\sin(2x)\,dx = -\frac{1}{2}x^2\cos(2x)\Big|_0^{\pi} + \int_0^{\pi} x\cos(2x)\,dx$

$u=x^2 \qquad dv=\sin(2x)dx$

$du=2x\,dx \qquad v=-\frac{1}{2}\cos(2x)$

$= -\frac{1}{2}x^2\cos(2x)+\frac{1}{2}\sin(2x)\cdot x\Big|_0^{\pi} - \frac{1}{2}\int_0^{\pi}\sin(2x)dx$

$u=x \qquad dv=\cos(2x)dx$

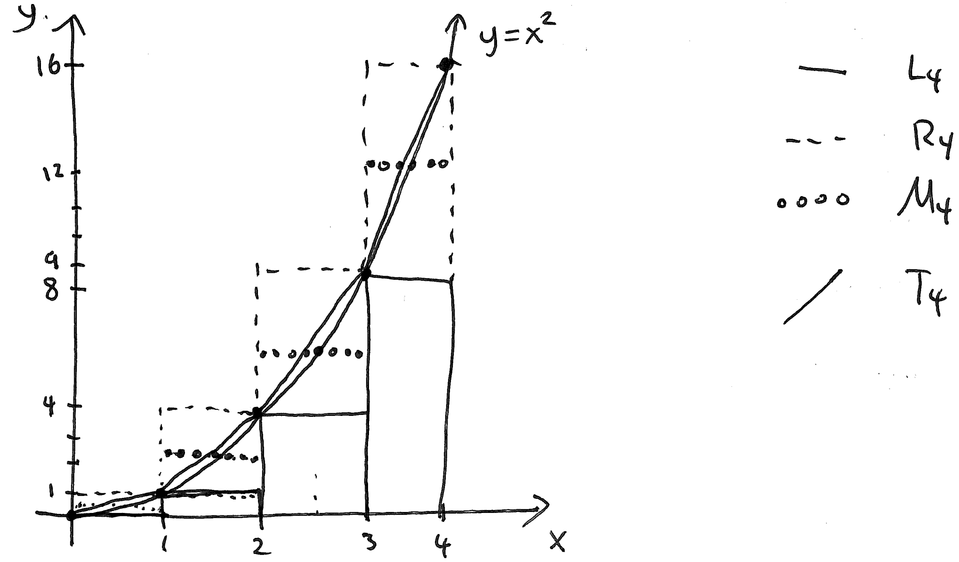
$du=dx \qquad v=\frac{1}{2}\sin(2x)$

$= -\frac{1}{2}x^2\cos(2x)+\frac{1}{2}\sin(2x)x+\frac{1}{4}\cos(2x)\Big|_0^{\pi}$

$= \left(-\frac{1}{2}\pi^2+0+\frac{1}{4}\right)-\left(0+0+\frac{1}{4}\right)$

$= \boxed{-\frac{\pi^2}{2}}$

2]



$$M_4 = \Delta x\left(f(\bar{x}_1)+f(\bar{x}_2)+f(\bar{x}_3)+f(\bar{x}_4)\right)$$
$$= 1\left(\left(\tfrac{1}{2}\right)^2+\left(\tfrac{3}{2}\right)^2+\left(\tfrac{5}{2}\right)^2+\left(\tfrac{7}{2}\right)^2\right)$$
$$= \underline{21}$$

$$T_4 = \frac{\Delta x}{2}\left(f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\right)$$
$$= \frac{1}{2}\left(0+2\cdot1^2+2\cdot2^2+2\cdot3^2+4^2\right)$$
$$= \underline{22}$$

3] Find smallest N such that $|E_{T_N}| < 0.0001$ for $\int_0^1 e^{-x^3}dx$: note typo in problem, should say $|f''(x)| \le 3$ on $[0,1]$.

Since $|f''(x)| \le 3$ on $[0,1]$, choose $K=3$: $\dfrac{3(1-0)^3}{12N^2} < .0001 \Rightarrow \dfrac{10000}{4} < N^2$

$N > \dfrac{100}{2} = \boxed{50}$

4] We have:

hr t	0	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{7}{12}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{5}{6}$	$\frac{11}{12}$	1
mph $v(t)$	550	575	600	580	610	640	625	595	590	620	640	640	630

. Goal: Approximate $D = \int_0^1 v(t)\,dt$.

$$S_{12} = \frac{\Delta t}{3}\left(v(0)+4v\left(\tfrac{1}{12}\right)+2v\left(\tfrac{1}{6}\right)+4v\left(\tfrac{1}{4}\right)+2v\left(\tfrac{1}{3}\right)+4v\left(\tfrac{5}{12}\right)+2v\left(\tfrac{1}{2}\right)+4v\left(\tfrac{7}{12}\right)+2v\left(\tfrac{2}{3}\right)+4v\left(\tfrac{3}{4}\right)+2v\left(\tfrac{5}{6}\right)+4v\left(\tfrac{11}{12}\right)+v(1)\right)$$
$$= \frac{\frac{1}{12}}{3}\left(550+4\cdot575+2\cdot600+4\cdot580+2\cdot610+4\cdot640+2\cdot625+4\cdot595+2\cdot590+4\cdot620+2\cdot640+4\cdot640+630\right)$$
$$\approx \boxed{608.61\,\text{mi}}$$

5] Make the substitution $u = 1+x$: then $du = dx$, $x=0 \Leftrightarrow u=1$
$x\to\infty \Leftrightarrow u\to\infty$

$\displaystyle\int_0^\infty \frac{dx}{(1+x)^{2q}} = \int_1^\infty \frac{du}{u^{2q}}$. This integral is a "p-integral", so it converges if $2q>1$, i.e. $\boxed{q>\frac{1}{2}}$